Basic Definitions

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Recall that:

- A **set** is a collection of elements.
- A **relation** R defined on elements of a set S is a subset of $S \times S$.
- An **equivalence relation** is symmetric, reflexive, and transitive.
- A **function** f from A to B is a subset of $A \times B$ such that for each $a \in A$, there is precisely one b such that $(a, b) \in f$. By convention, we denote $f(a) = b$.
- A **partially ordered set**, or **poset**, is a set with a relation $\leq$ satisfying the properties of reflexivity, anticommutativity, and transitivity.
- A **group** G is a set with an operation on its elements, $*$, that satisfies the following properties:
  - $*$ is associative.
  - There is an element e for which $e*g = g$ and $g*e = g$ for any element $g \in G$.
  - For any $g \in G$ there exists a (unique) element g^{-1} for which $g*g^{-1} = e$ and $g^{-1} * g = e$.
  - An **abelian group** is a group where $*$ is commutative.
- A ring is a set R with two operations $+$ and $*$, so that:
  - R is an abelian group under $+$.
  - $*$ is associative.
  - The operations satisfy the distributive property $a*(b+c) = a*b+a*c$.
  - Most rings we encounter will have a multiplicative identity, known as 1. Some authors take this as a requirement for a ring. Note that the set of even integers is a ring without an identity element, sometimes referred to as a "rng". ⌣̈
- Some other properties a ring may have are:

- A ring R where $a * b = b * a$ for all $a, b \in R$ is called a **commutative ring**.
- A nonzero ring R where $a * b = 0$ implies that either a or b is zero is called an **domain**. If it is commutative, it is called an **integral domain**.
- A **unit** is an element of a ring invertible with respect to $*$. If every element is invertible, R is a **division ring**.
- A commutative division ring is known as a **field**.

* A **left R-module** over a ring R with unit consists of an abelian group M under $+$, and an operation $R \times M \to M$, known as **scalar multiplication**, for which:
  - $r(x + y) = rx + ry$
  - $(r + s)x = rx + sx$
  - $(rs)x = r(sx)$
  - $1_R x = x$.

  A right R-module is defined similarly.